Journal of Approximation Theory **94**, 54–72 (1998)
Article No. AT983177

On a Generalized Best Approximation Problem

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Communicated by Frank Deutsch

Received April 17, 1995; accepted in revised form June 4, 1997

Let C be a closed bounded convex subset of a Banach space E which has the origin of E as an interior point and let p_C denote the Minkowski functional with respect to C. Given a closed set $X \subset E$ and a point $u \in E$ we consider a minimization problem $\min_C(u, X)$ which consists in proving the existence of a point $\tilde{x} \in X$ such that $p_C(\tilde{x} - u) = \lambda_C(u, X)$, where $\lambda_C(u, X) = \inf\{p_C(x-u) \mid x \in X\}$. If such a point is unique and every sequence $\{x_n\} \subset X$ satisfying the condition $\lim_{n \to +\infty} p_C(x_n - u) = \lambda_C(u, X)$ converges to this point, the minimization problem $\min(u, X)$ is called well posed. Under the assumption that the modulus of convexity with respect to p_C is strictly positive, we prove that for every closed subset X of E, the set $E_o(X)$ of all $u \in E$ for which the minimization problem $\min_C(u, X)$ is well posed is a residual subset of E. In fact we show more, namely that the set $E \backslash E_o(X)$ is σ-porous in E. Moreover, we prove that for most closed bounded subsets X of E, the set $E \backslash E_o(X)$ is dense in E. © 1998 Academic Press

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper E is a Banach space of dimension at least 2. For $X \subset E$ ($X \neq \emptyset$) by int X, ∂X, and diam X we mean the interior of X, the boundary of X, and the diameter of X, respectively. If $x, y \in E$, xy stands for the closed interval with end points x and y. A closed ball in E with center x and radius $r > 0$ is denoted by $S(x, r)$. For notational convenience we put $S = S(0, 1)$.

Define

$$\mathfrak{B} = \{X \subset E \mid X \text{ is nonempty closed bounded}\}.$$

0021-9045/98 $25.00

We suppose $\mathfrak{B}$ is endowed with the Hausdorff metric. As is well known under this metric $\mathfrak{B}$ is a complete metric space.

Throughout this paper C will denote a closed bounded convex subset of E with $0 \in \operatorname{int} C$. Clearly C is an absorbing not necessarily symmetric subset of E. Recall that the functional of Minkowski $p_C: E \to \mathbb{R}$ with respect to the set C is defined by

$$p_C(x) = \inf \{\alpha > 0 \mid x \in \alpha C\}. \tag{1.1}$$

For $X \in \mathfrak{B}$ and $u \in E$ put

$$\lambda_C(u, X) = \inf \{ p_C(x-u) \mid x \in X\}. \tag{1.2}$$

It is easy to see that $\lambda_C(u, X)$ is continuous as a function of $u \in E$.

Given $X \in \mathfrak{B}$ and $u \in E$ let us consider the minimization problem, denoted by

$$\min_C (u, X) \tag{1.3}$$

which consists in finding points $\tilde{x} \in X$ (if they exist) satisfying $p_C(\tilde{x} - u) = \lambda_C(u, X)$. Any such point $\tilde{x}$ is called a *solution* of (1.3) and any sequence $\{x_n\} \subset X$ such that $\lim_{n \to +\infty} p_C(x_n - u) = \lambda_C(u, X)$ is called a *minimizing sequence* of the minimization problem (1.3). The problem (1.3) is said to be *well posed* if it has a unique solution, say x_o, and every minimizing sequence converges to x_o.

Let $\delta_C: [0, 2] \to [0, +\infty)$ be the *modulus of convexity* of C, i.e.,

$$\delta_C(\varepsilon) = \inf \left\{ 1 - p_C\left(\frac{x+y}{2}\right) \,\middle|\, x, y \in C \text{ and } p_C(x-y) \geqslant \varepsilon \right\}. \tag{1.4}$$

Note that the function δ_C is well defined, nondecreasing, $\delta_C(0) = 0$, and $\delta_C(2) \leqslant 1$.

Supposing that $\delta_C(\varepsilon) > 0$ for each $\varepsilon \in (0, 2]$, we will prove that for every closed subset X of E the set $E_o(X)$ of all $u \in E$ for which the problem (1.3) is well posed is a residual subset of E. In fact we will prove more, namely that the set $E \backslash E_o(X)$ is σ-porous in E. Moreover we will show that for most (in the sense of the Baire category) closed bounded subsets X of E the set $E \backslash E_o(X)$ is dense in E.

In the present paper we generalize some results from [5–7, 18, 19]. Further results in the same spirit can be found in [7–11, 14, 19]. A comprehensive investigation of various moduli of convexity for sets C can be found in [12, 13, 20]. The last three papers were brought to our attention while correcting the galley proofs.

2. AUXILIARY RESULTS

We start with some well known properties of the Minkowski functional which follow immediately from the definition.

PROPOSITION 2.1. *Let E and C be as above. Then for every $x, x' \in E$ we have*

(i) $p_C(x) \geqslant 0$;
(ii) $p_C(x+x') \leqslant p_C(x) + p_C(x')$;
(iii) $p_C(\lambda x) = \lambda p_C(x)$, *if* $\lambda \geqslant 0$;
(iv) $p_C(x) = 1$ *iff* $x \in \partial C$;
(v) $p_C(x) < 1$ *iff* $x \in \operatorname{int} C$;
(vi) $p_C(x) > 1$ *iff* $x \notin C$;
(vii) $p_C(x) = 0$ *iff* $x = 0$;
(viii) $p_{\lambda C}(x) = (1/\lambda)\, p_C(x)$ *if* $\lambda > 0$.

For the reader's convenience we recall also the following elementary

LEMMA 2.1. *Let $f\colon [0, 2] \to [0, 1]$ be a convex function. Then for every $x, y, u, v \in [0, 2]$ such that $x < y \leqslant v$ and $x \leqslant u < v$ we have*

$$\frac{f(y) - f(x)}{y - x} \leqslant \frac{f(v) - f(u)}{v - u}.$$

Moreover, if f is also nondecreasing, for every $0 < \alpha < 2$ the function f restricted to $[0, \alpha]$ is lipschitzian with constant $L = 1/(2 - \alpha)$.

PROPOSITION 2.2. *The function δ_C given by (1.4) is continuous in the interval $[0, 2)$.*

Proof. For $u, v \in E$ with $u \neq 0$ and $p_C(u) \geqslant p_C(-u)$ set

$$A(u, v) = \{(x, y) \mid x, y \in C, x - y = \alpha u \text{ and } x + y = \beta v \text{ for some } \alpha \geqslant 0, \beta \geqslant 0\}.$$

Now for $\varepsilon \in [0, 2]$ set

$$\delta_C(u, v; \varepsilon) = \inf\left\{1 - p_C\left(\frac{x+y}{2}\right) \,\middle|\, (x, y) \in A(u, v) \text{ and } p_C(x - y) \geqslant \varepsilon\right\}. \tag{2.1}$$

Observe that the number $\delta_C(u, v; \varepsilon)$ is well defined. For this, it suffices to show that the set in brackets is nonempty. Indeed, given $u, v \in E$ $(u \neq 0, p_C(u) \geqslant p_C(-u))$ and $\varepsilon \in [0, 2]$ take $x = u/p_C(u)$ and put $y = -x$. Clearly $(x, y) \in A(u, v)$ and $p_C(x-y) = 2 \geqslant \varepsilon$. Further $\delta_C(u, v; 0) = 0$ and $\delta_C(u, v; \cdot)$ is nondecreasing.

Claim 1. $\delta_C(u, v; \cdot)$ is convex in the interval $[0,2]$.

Indeed, let $\varepsilon_1, \varepsilon_2 \in [0, 2]$, $\varepsilon_1 < \varepsilon_2$, and $\lambda \in [0, 1]$. Given $\sigma > 0$ choose $(x_i, y_i) \in A(u, v)$ with $p_C(x_i - y_i) \geqslant \varepsilon_i$, $i = 1, 2$, such that

$$\delta_C(u, v; \varepsilon_i) > 1 - p_C\left(\frac{x_i + y_i}{2}\right) - \sigma. \tag{2.2}$$

Put $x_3 = \lambda x_1 + (1-\lambda)\, x_2$ and $y_3 = \lambda y_1 + (1-\lambda)\, y_2$. It is easy to see that $(x_3, y_3) \in A(u, v)$. Let $\alpha_i, \beta_i \geqslant 0$ be such that

$$x_i - y_i = \alpha_i u \qquad \text{and} \qquad x_i + y_i = \beta_i v, \qquad i = 1, 2, 3.$$

Since $\alpha_i p_C(u) = p_C(\alpha_i u) = p_C(x_i - y_i) \geqslant \varepsilon_i$, we have

$$\begin{aligned} p_C(x_3 - y_3) &= p_C(\lambda\alpha_1 u + (1-\lambda)\,\alpha_2 u) = (\lambda\alpha_1 + (1-\lambda)\,\alpha_2)\, p_C(u) \\ &= \lambda p_C(\alpha_1 u) + (1-\lambda)\, p_C(\alpha_2 u) \geqslant \lambda\varepsilon_1 + (1-\lambda)\,\varepsilon_2 \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} p_C(x_3 + y_3) &= p_C(\lambda\beta_1 v + (1-\lambda)\,\beta_2 v) = (\lambda\beta_1 + (1-\lambda)\,\beta_2)\, p_C(v) \\ &= \lambda p_C(\beta_1 v) + (1-\lambda)\, p_C(\beta_2 v) \\ &= \lambda p_C(x_1 + y_1) + (1-\lambda)\, p_C(x_2 + y_2). \end{aligned} \tag{2.4}$$

Now, by virtue of (2.3), (2.1), (2.4), and (2.2) we have

$$\begin{aligned} &\delta_C(u, v; \lambda\varepsilon_1 + (1-\lambda)\,\varepsilon_2) \\ &\quad \leqslant 1 - p_C\left(\frac{x_3 + y_3}{2}\right) \\ &\quad = 1 - \frac{\lambda}{2}\, p_C(x_1 + y_1) - \frac{1-\lambda}{2}\, p_C(x_2 + y_2) \\ &\quad = \lambda\left(1 - p_C\left(\frac{x_1 + y_1}{2}\right)\right) + (1-\lambda)\left(1 - p_C\left(\frac{x_2 + y_2}{2}\right)\right) \\ &\quad < \lambda\delta_C(u, v; \varepsilon_1) + (1-\lambda)\,\delta_C(u, v; \varepsilon_2) + \sigma. \end{aligned}$$

As $\sigma > 0$ is arbitrary, it follows that

$$\delta_C(u, v; \lambda\varepsilon_1 + (1-\lambda)\,\varepsilon_2) \leqslant \lambda\delta_C(u, v; \varepsilon_1) + (1-\lambda)\,\delta(u, v; \varepsilon_2)$$

which completes the proof of Claim 1.

Claim 2. For every $\varepsilon \in [0, 2]$ we have

$$\delta_C(\varepsilon) = \inf\{\delta_C(u, v; \varepsilon) \mid u, v \in E, u \neq 0 \text{ and } p_C(u) \geqslant p_C(-u)\}.$$

Clearly $\delta_C(\varepsilon) \leqslant \delta_C(u, v; \varepsilon)$. Therefore to prove the claim it suffices to show that for every $x, y \in C$ with $p_C(x-y) \geqslant \varepsilon$ there are $u, v \in E$ with $u \neq 0$ and $p_C(u) \geqslant p_C(-u)$, such that

$$\delta_C(u, v; \varepsilon) \leqslant 1 - p_C\left(\frac{x+y}{2}\right). \tag{2.5}$$

Indeed, let $x, y \in C$ with $p_C(x-y) \geqslant \varepsilon$. If $p_C(x-y) \geqslant p_C(y-x)$, put $u = x-y$ and $v = x+y$. Clearly $(x, y) \in A(u, v)$ and the relation (2.5) follows immediately from (2.1). If $p_C(x-y) < p_C(y-x)$, put $u = y-x$ and $v = y+x$. Clearly $(y, x) \in A(u, v)$ and $p_C(y-x) \geqslant \varepsilon$. The relation (2.5) follows again from (2.1). This completes the proof of Claim 2.

By Lemma 2.1 every function $\delta_C(u, v; \cdot)$ $(u \neq 0, p_C(u) \geqslant p_C(-u))$ restricted to $[0, \alpha]$, $0 < \alpha < 2$, is lipschitzian with constant $1/(2-\alpha)$ and so, δ_C restricted to $[0, \alpha]$ is Lipschitzian with the same constant. This completes the proof of Proposition 2.2.

Define

$$\varepsilon_o = \sup\{\varepsilon \geqslant 0 \mid \delta_C(\varepsilon) = 0\}. \tag{2.6}$$

PROPOSITION 2.3. *The function* δ_C *given by* (1.4) *is strictly increasing in the interval* $[\varepsilon_0, 2]$, *provided* $\varepsilon_0 < 2$.

Proof. Suppose for a contradiction then there is $\sigma > \varepsilon_o$ and $\theta > 0$, $\sigma + \theta \leqslant 2$, such that $\delta_C(\sigma+\theta) = \delta_C(\sigma)$. Let $\eta > 0$ be arbitrary. By virtue of Claim 2 of Proposition 2.2 there are $u, v \in E$ $(u \neq 0, p_C(u) \geqslant p_C(-u))$ such that

$$\delta_C(\sigma+\theta) \geqslant \delta_C(u, v; \sigma+\theta) - \eta. \tag{2.7}$$

Since $\delta_C(\sigma) \leqslant \delta_C(u, v; \sigma)$ and $\delta_C(u, v; \cdot)$ is convex, by virtue of Lemma 2.1 we have

$$\frac{\delta_C(\sigma)}{\sigma} \leqslant \frac{\delta_C(u, v; \sigma)}{\sigma} = \frac{\delta_C(u, v; \sigma) - \delta_C(u, v; 0)}{\sigma}$$

$$\leqslant \frac{\delta_C(u, v; \sigma + \theta) - \delta_C(u, v; \sigma)}{\sigma}.$$

By the last inequality and (2.7) we have $\delta_C(\sigma) \leqslant \eta$. Since $\eta > 0$ is arbitrary, it follows that $\delta_C(\sigma) = 0$. This is a contradiction, because $\sigma > \varepsilon_o$. The proof of Proposition 2.3 is complete.

A simple calculation shows that for $r > 0$ we have

$$p_C(x+y) \leqslant 2r\left(1 - \delta_C\left(\frac{p_C(x-y)}{r}\right)\right) \qquad \text{for every} \quad x, y \in rC. \tag{2.8}$$

Define $\delta_C^*\colon [0, 1] \to \mathbb{R}$ by

$$\delta_C^*(\sigma) = \begin{cases} \varepsilon_o, & \text{if} \quad \sigma = 0, \\ \delta_C^{-1}(\sigma), & \text{if} \quad 0 < \sigma < \delta_C(2), \\ 2, & \text{if} \quad \delta_C(2) \leqslant \sigma \leqslant 1, \end{cases} \tag{2.9}$$

where ε_o is given by (2.6).

Note that δ_C^* is a continuous nondecreasing function and $\delta_C^*(0) = 0$ provided $\delta(\varepsilon) > 0$ for every $\varepsilon \in (0, 2]$.

Proposition 2.4. *Suppose that $\delta_C(\varepsilon) > 0$ for every $\varepsilon \in (0, 2]$. Let $x, y \in E \backslash \{0\}$. Then*

$$p_C(x) + p_C(y) = p_C(x+y) \tag{2.10}$$

if and only if $y = \lambda x$ for some $\lambda \geqslant 0$.

Proof. A simple calculation shows that if $y = \lambda x$ with $\lambda \geqslant 0$, then (2.10) holds. Suppose now that (2.10) holds for some $x, y \in E \backslash \{0\}$. Let $\tilde{x}, \tilde{y} \in \partial C$ be such that $x = \alpha\tilde{x}$, $y = \beta\tilde{y}$, $\alpha > 0$, $\beta > 0$. Suppose $\tilde{x} \neq \tilde{y}$. Taking $\tilde{\varepsilon} = \min\{p_C(\tilde{x} - \tilde{y}), p_C(\tilde{y} - \tilde{x})\}$, by virtue of (1.4), we have

$$p_C(\tilde{x} + \tilde{y}) \leqslant 2(1 - \delta_C(\tilde{\varepsilon})).$$

Without loss of generality we can suppose that $\alpha \leqslant \beta$. By virtue of Proposition 2.1, the relation (2.10), and the last inequality we have

$$\begin{aligned}\alpha+\beta &= p_C(\alpha\tilde{x})+p_C(\beta\tilde{y})=p_C(\alpha\tilde{x}+\beta\tilde{y})\\ &= p_C(\alpha\tilde{x}+\alpha\tilde{y}+(\beta-\alpha)\ \tilde{y})\leqslant \alpha p_C(\tilde{x}+\tilde{y})+(\beta-\alpha)\\ &\leqslant 2\alpha(1-\delta_C(\tilde{\varepsilon}))+\beta-\alpha=\alpha+\beta-2\alpha\delta_C(\tilde{\varepsilon}),\end{aligned}$$

a contradiction. Thus $\tilde{x}=\tilde{y}$ and so $y=(\beta/\alpha)\,x$, which completes the proof.

PROPOSITION 2.5. *Let* $u, v\in\partial(C+x)$, *where* $x\in E$ *and* $u\neq v$. *Then for every* $t\in(0,1)$ *we have*

$$p_C(u-y_t)<p_C(v-y_t),$$

where $y_t=tx+(1-t)\,u$.

Proof. Suppose for a contradiction that for some $t\in(0,1)$ we have

$$p_C(v-y_t)\leqslant p_C(u-y_t).$$

Clearly

$$\begin{aligned}1=p_C(v-x)&\leqslant p_C(v-y_t)+p_C(y_t-x)\\ &\leqslant p_C(u-y_t)+p_C(y_t-x)=p_C(u-x)=1.\end{aligned}$$

Thus

$$p_C(v-y_t)+p_C(y_t-x)=p_C(v-x),$$

and so, by Proposition 2.4, $y_t-x=\lambda(v-y_t)$ for some $\lambda>0$, a contradiction. This completes the proof.

For $X\in\mathfrak{B}$ and $u\in E$ set

$$\Lambda_C(u,X)=\sup\{p_C(x-u)\mid x\in X\}.$$

LEMMA 2.2. *Supose that* $\delta_C(\varepsilon)>0$ *for every* $\varepsilon\in(0,2]$. *Let* $x\in E$ *and* $r>0$. *Let* $y\in E$, $y\neq x$, *be such that* $p_C(y-x)\leqslant r/2$. *Then for every* $0<\sigma<2p_C(y-x)$ *we have*

$$\Lambda_C(\tilde{y}, D_C(x,y;r,\sigma))\leqslant\sigma+(r-p_C(y-x))\,\delta_C^*\left(\frac{\sigma}{2p_C(y-x)}\right), \tag{2.11}$$

where δ_C^* *is given by* (2.9),

$$\tilde{y}=y+(r-p_C(y-x))\,\frac{y-x}{p_C(y-x)}, \tag{2.12}$$

and

$$D_C(x, y; r, \sigma) = [y + (r - p_C(y-x) + \sigma)\, C] \backslash (x + r \operatorname{int} C).$$

Proof. Let x, y, r, σ, and $\tilde{y}$ be as above. Let $z \in D_C(x, y; r, \sigma)$. Set

$$\tilde{z} = y + (r - p_C(y-x))\,\frac{z-y}{p_C(z-y)}. \tag{2.13}$$

Suppose that $p_C(z - \tilde{y}) > \sigma$. By (2.13) and the inequality $p_C(z-y) \leqslant r - p_C(y-x) + \sigma$ we have

$$\begin{aligned} p_C(z-\tilde{z}) &= p_C((z-y) - (\tilde{z}-y)) \\ &= p_C\left((z-y) - (r - p_C(y-x))\,\frac{z-y}{p_C(z-y)}\right) \\ &= p_C(z-y) - r + p_C(y-x) \leqslant \sigma. \end{aligned} \tag{2.14}$$

Thus

$$p_C(\tilde{z}-\tilde{y}) = p_C((z-\tilde{y}) - (z-\tilde{z})) \geqslant p_C(z-\tilde{y}) - p_C(z-\tilde{z}) > 0.$$

Now using (2.12) we have

$$\begin{aligned} z - x &= (z-\tilde{z}) + (\tilde{z}-y) + (y-x) \\ &= (z-\tilde{z}) + (\tilde{z}-y) + \frac{p_C(y-x)}{r-p_C(y-x)}\,(\tilde{y}-y) \\ &\quad - \frac{p_C(y-x)}{r-p_C(y-x)}\,(\tilde{z}-y) + \frac{p_C(y-x)}{r-p_C(y-x)}\,(\tilde{z}-y) \\ &= (z-\tilde{z}) + \left(1 - \frac{p_C(y-x)}{r-p_C(y-x)}\right)(\tilde{z}-y) \\ &\quad + \frac{p_C(y-x)}{r-p_C(y-x)}\,[(\tilde{y}-y) + (\tilde{z}-y)]. \end{aligned}$$

From this, by virtue of Proposition 2.1 and (2.14), (2.13), (2.8) we have

$$p_C(z-x) \leqslant \sigma + \left(1 - \frac{p_C(y-x)}{r - p_C(y-x)}\right)(r - p_C(y-x))$$
$$+ \frac{p_C(y-x)}{r-p_C(y-x)} 2(r - p_C(y-x)) \left(1 - \delta_C\left(\frac{p_C(\tilde{y}-\tilde{z})}{r-p_C(y-x)}\right)\right)$$
$$= \sigma + r - 2p_C(y-x)\,\delta_C\left(\frac{p_C(\tilde{y}-\tilde{z})}{r-p_C(y-x)}\right),$$

and, since $p_C(z-x) \geqslant r$, we have

$$\delta_C\left(\frac{p_C(\tilde{y}-\tilde{z})}{r-p_C(y-x)}\right) \leqslant \frac{\sigma}{2p_C(y-x)}.$$

From the last inequality, Proposition 2.3, and the definition (2.9) it follows that

$$\frac{p_C(\tilde{y}-\tilde{z})}{r-p_C(y-x)} \leqslant \delta_C^*\left(\frac{\sigma}{2p_C(y-x)}\right). \tag{2.15}$$

By the inequality $p_C(z-\tilde{y}) \leqslant p_C(z-\tilde{z}) + p_C(\tilde{z}-\tilde{y})$ and the relations (2.14) and (2.15) we have

$$p_C(z-\tilde{y}) \leqslant \sigma + (r - p_C(y-x))\,\delta_C^*\left(\frac{\sigma}{2p_C(y-x)}\right). \tag{2.16}$$

The last inequality proved for $z \in D_C(x, y; r, \sigma)$ with $p_C(z-\tilde{y}) > \sigma$ is trivially satisfied if $p_C(z-\tilde{y} \leqslant \sigma$. Thus (2.16) is true for every z in $D_C(x, y; r, \sigma)$, whence the statement of Lemma 2.2 follows.

3. EXISTENCE

Let E, C, $\mathfrak{B}$, and S be as in Section 1. Set

$$\mu = \inf_{x \in \partial S} p_C(x) \qquad \text{and} \qquad \nu = \sup_{x \in \partial S} p_C(x). \tag{3.1}$$

Note that $0 < \mu \leqslant \nu < +\infty$ and that for every $x \in E$ we have

$$\mu\,\|x\| \leqslant p_C(x) \leqslant \nu\,\|x\|. \tag{3.2}$$

Given $X \in \mathfrak{B}$, $u \in E$, and $\sigma > 0$ define

$$L_C(u, X; \sigma) = \{x \in X \mid p_C(x-u) \leqslant \lambda_C(u, X) + \sigma\}.$$

PROPOSITION 3.1. *Let $X \in \mathfrak{B}$ and $u \in E$ be given. Then the problem* (1.3) *is well posed if and only if*

$$\inf_{\sigma > 0} \operatorname{diam} L_C(u, X; \sigma) = 0.$$

Proof. This is similar to that of [5, Proposition 2].

LEMMA 3.1. *Under the hypotheses and with the same notations of Lemma* 2.2 *we have*

$$\operatorname{diam} D_C(x, y; r, \sigma) \leqslant \omega_C(x, y; r, \sigma), \tag{3.3}$$

where

$$\omega_C(x, y; r, \sigma) = \frac{2}{\mu}\left(\sigma + (r - p_C(y - x))\, \delta_C^*\left(\frac{\sigma}{2p_C(y - x)}\right)\right). \tag{3.4}$$

Proof. In view of (3.2) and the definition of Λ_C, for every $z \in D_C(x, y; r, \sigma)$ we have

$$\mu \, \|z - \tilde{y}\| \leqslant p_C(z - \tilde{y}) \leqslant \Lambda_C(\tilde{y}, D_C(x, y; r, \sigma)).$$

From this and Lemma 2.2 the statement follows.

Remark 3.1. Under the hypotheses of Lemma 2.2, from Propositions 2.2 and 2.3 and the definition (2.9) it follows that the function $\omega_C(x, y; r, \cdot)$ is well defined, continuous and strictly increasing in the interval $[0, 2p_C(y - x)]$. Clearly $\omega_C(x, y; r, 0) = 0$ and $\omega_C(x, y; r, 2p_C(y - x)) = 4r/\mu$.

Let $\omega_C^{-1}(x, y; r, \cdot)$ denote the inverse function of $\omega_C(x, y; r, \cdot)$, defined in the interval $[0, 4r/\mu]$.

THEOREM 3.1. *Let E and C be as in Section* 1. *Suppose that $\delta_C(\varepsilon) > 0$ for $\varepsilon \in (0, 2]$. Let X be a nonempty closed subset of E. Denote by E^o the set of all $u \in E$ such that the minimization problem $\min_C(u, X)$ is well posed. Then E^o is a dense G_δ subset of E.*

Proof. For $k \in \mathbb{N}$ set

$$E_k = \{u \in E \mid \inf_{\sigma > 0} \operatorname{diam} L_C(u, X; \sigma) < \varepsilon_k\},$$

where $\varepsilon_k = 1/2^k$.

Claim 1. E_k is dense in E.

Indeed, let $u \in E \backslash X$ (if $u \in X$ there is nothing to prove) and let $0 < r < \lambda_C(u, X)/2$. Let $0 < r' < \min\{r, \mu r\}$, where μ is given by (3.1). By virtue of Lemma 3.1 and Remark 3.1 there is $\sigma_0 > 0$ such that, for every $y \in E$ satisfying $p_C(y-u) = r'$, we have

$$\operatorname{diam} D_C(u, y; \lambda_C(u, X), 2\sigma_0) < \varepsilon_k. \tag{3.5}$$

Let $x \in X$ be such that $p_C(x-u) < \lambda_C(u, X) + \sigma_0$. Let $y \in xu$ be such that $p_C(y-u) = r'$. A simple calculation shows that

$$\lambda_C(y, X) < \lambda_C(u, X) - p_C(y-u) + \sigma_o.$$

Using the last inequality, it is easy to verify that

$$L_C(y, X; \sigma_0) \subset D_C(u, y; \lambda_C(u, X), 2\sigma_0). \tag{3.6}$$

From (3.6) and (3.5) it follows that $y \in E_k$. To complete the proof it suffices to note that

$$\|y-u\| \leqslant \frac{1}{\mu} p_C(y-u) = \frac{r'}{\mu} < r.$$

Claim 2. E_k is open in E.

Indeed, let $u \in E_k$. Let $\sigma_0 > 0$ be such that

$$\operatorname{diam} L_C(u, X; \sigma_0) < \varepsilon_k. \tag{3.7}$$

Let $0 < \delta \leqslant \sigma_0/(1+2v)$, where v is given by (3.1). We will prove that $S(u, \delta) \subset E_k$. In fact, let $y \in S(u, \delta)$ and let $x \in L_C(y, X; \delta)$ be arbitrary. By Proposition 2.1(ii), the relation (3.2), and the choice of δ we have

$$\begin{aligned} p_C(x-u) &\leqslant p_C(x-y) + p_C(y-u) \leqslant \lambda_C(y, X) + \delta + p_C(y-u) \\ &\leqslant \lambda_C(u, X) + p_C(u-y) + \delta + p_C(y-u) \\ &\leqslant \lambda_C(u, X) + 2v\,\|y-u\| + \delta \leqslant \lambda_C(u, X) + (2v+1)\,\delta \\ &\leqslant \lambda_C(u, X) + \sigma_0. \end{aligned}$$

Since x is arbitrary in $L_C(y, X; \delta)$ it follows that $L_C(y, X; \delta) \subset L_C(u, X; \sigma_0)$, which by virtue of (3.7) implies $y \in E_k$. Consequently $S(u, \delta) \subset E_k$. This completes the proof of Claim 2.

Now set $\tilde{E} = \bigcap_{k=1}^{\infty} E_k$. Using the Proposition 3.1 it is easy to show that $\tilde{E} = E_0$, whence the statement of Theorem 3.1 follows.

4. POROSITY

A subset X of E is said to be *porous* in E if there exist $0 < \alpha \leqslant 1$ and $r_0 > 0$ such that for every $x \in E$ and $r \in (0, r_0]$ there is a point $y \in E$ such that $S(y, \alpha r) \subset S(x, r) \cap (E \setminus X)$. A subset X of E is called *σ-porous* in E if it is a countable union of sets which are porous in E. Note that in the definition of a porous set the statement "for every $x \in E$" can be replaced by "for every $x \in X$."

Clearly, a set which is σ-porous in E is also merger in E, the converse being false, in general. Furthermore, if $E = \mathbb{R}^n$, then each σ-porous set has (Lebesgue) measure zero.

LEMMA 4.1. *Let E and C be as in Section* 1 *and let μ and v be given by* (3.1). *Let $X \in \mathfrak{B}$ and $z \in E \setminus X$. Suppose that the problem* $\min_C(z, X)$ *has a unique solution, say x_0. Let $I_z = z\hat{z}$, where $\hat{z} = (1/2)(x_0 + z)$. Let $0 < \varepsilon < 4\lambda_C(z, X)/\mu$ and let $y \in I_z$. Define*

$$\rho_{yz}(\varepsilon) = \min \left\{ \frac{1}{1+2v}\, \omega_C^{-1}(z, y; \lambda_C(z, X), \varepsilon), \frac{1}{1+2v}\, p_C(y - z) \right\}. \tag{4.1}$$

Then

$$\operatorname{diam} L_C(u, X; \rho_{yz}(\varepsilon)) \leqslant \varepsilon \qquad \textit{for every} \quad u \in S(y, \rho_{yz}(\varepsilon)).$$

Proof. Let $z \in E \setminus X$, $y \in I_z$, and $\varepsilon > 0$ satisfy the hypotheses of Lemma 4.1. Set $\rho_o = \rho_{yz}(\varepsilon)$. Let $u \in S(y, \rho_0)$ be arbitrary. We will prove that

$$L_C(u, X; \rho_o) \subset L_C(y, X; (1+2v)\, \rho_o). \tag{4.2}$$

Indeed, let $x \in L_C(u, X; \rho_o)$. We have

$$\begin{aligned} p_C(x-y) &\leqslant p_C(x-u) + p_C(u-y) \leqslant \lambda_C(u, X) + \rho_o + p_C(u-y) \\ &\leqslant \lambda_C(y, X) + p_C(y-u) + \rho_o + p_C(u-y) \\ &\leqslant \lambda_C(y, X) + 2v\, \|u-y\| + \rho_o \leqslant \lambda_C(y, X) + (1+2v)\, \rho_o, \end{aligned}$$

whence (4.2) follows.

Furthermore, since $\lambda_C(y, X) = \lambda_C(z, X) - p_C(y-z)$, we have

$$\begin{aligned} &L_C(y, X; (1+2v)\, \rho_o) \\ &\quad = \{x \in X \mid p_C(x-y) \leqslant \lambda_C(y, X) + (1+2v)\, \rho_o\} \\ &\quad = \{x \in X \mid p_C(x-y) \leqslant \lambda_C(z, X) - p_C(y-z) + (1+2v)\, \rho_o\} \\ &\quad \subset D_C(z, y; \lambda_C(z, X), (1+2v)\, \rho_o). \end{aligned} \tag{4.3}$$

Note that

$$0 < p_C(y-z) \leqslant \tfrac{1}{2}\lambda_C(z, X) \qquad \text{and} \qquad 0 < (1+2\nu)\,\rho_o < 2p_C(y-z). \tag{4.4}$$

By virtue of (4.2), (4.3), (4.4), and Lemma 3.1 we have

$$\begin{aligned}\operatorname{diam} L_C(u, X, \rho_o) &\leqslant \operatorname{diam} D_c(z, y; \lambda_C(z, X), (1+2\nu)\,\rho_o)\\ &\leqslant \omega_C(z, y; \lambda_C(z, X), (1+2\nu)\,\rho_o).\end{aligned} \tag{4.5}$$

By (4.1) we have

$$(1+2\nu)\,\rho_o \leqslant \omega_C^{-1}(z, y; \lambda_C(z, X), \varepsilon). \tag{4.6}$$

Since in the interval $[0, 2p_C(y-z)]$ the function $\omega_C(z, y; \lambda_C(z, X), \cdot)$ is strictly increasing (see Remark 3.1) from (4.6) it follows that

$$\omega_C(z, y; \lambda_C(z, X), (1+2\nu)\,\rho_o) \leqslant \varepsilon.$$

From this and (4.5) the statement of Lemma 4.1 follows.

THEOREM 4.1. *Under the hypotheses of Theorem* 3.1 *the set* $E \backslash E^o$ *is* σ*-porous in* E.

Proof. For $k \in \mathbb{N}$ set $\varepsilon_k = 1/2^k$. Define

$$\tilde{E} = \bigcap_{k \in \mathbb{N}} \bigcup_{z \in E^o} \bigcup_{y \in I_z} S(y, \rho_{yz}(\varepsilon_k)),$$

where I_z and $\rho_{yz}(\varepsilon_k)$ are as in Lemma 4.1 if $z \in E \backslash X$ while, $I_z = \{z\}$ and $\rho_{zz}(\varepsilon_k) = \varepsilon_k/2\nu$ if $z \in X$, and $\mathbb{N}$ stands for the set of all strictly positive integers.

Using Lemma 4.1 and Proposition 3.1 it is easy to see that $\tilde{E} \subset E^o$. Thus

$$E \backslash E^o \subset E \backslash \tilde{E} = \bigcup_{k \in \mathbb{N}} E_k = \bigcup_{k \in \mathbb{N}} \bigcup_{l \in \mathbb{N}} E_{kl},$$

where

$$E_k = E \backslash \bigcup_{z \in E^o} \bigcup_{y \in I_z} S(y, \rho_{yz}(\varepsilon_k)) \qquad \text{and}$$

$$E_{kl} = \left\{ z \in E_k \;\middle|\; \frac{1}{l} < \lambda_C(z, X) < l \right\}.$$

To complete the proof it suffices to show that for every $k, l \in \mathbb{N}$ the set E_{kl} is porous in E. Let $k, l \in \mathbb{N}$ be arbitrary. Define

$$r_o = \min\left\{\frac{1}{2l}, \frac{1}{2\nu l}\right\} \tag{4.7}$$

and

$$\alpha = \min\left\{\frac{1}{4}, \frac{\mu}{6}, \frac{\mu\alpha_o}{1+2\nu}, \frac{\mu\varepsilon_k}{2(1+2\nu)}\right\}, \tag{4.8}$$

where μ, ν are given by (3.1) and $\alpha_o \in (0, 1/2)$ is such that

$$\delta_C^*(\alpha_o) \leqslant \frac{\mu\varepsilon_k}{4l}. \tag{4.9}$$

We will show that the set E_{kl} is porous in E with r_0 and α given by (4.7) and (4.8).

Let $u \in E_{kl}$ and $0 < r \leqslant r_o$ be any. By virtue of Theorem 3.1 there is $z \in E^o$ such that

$$\|z-u\| < \frac{r}{4} \qquad \text{and} \qquad \frac{1}{l} < \lambda_C(z, X) < l.$$

Let $x_o \in X$ satisfy $p_C(x_o - z) = \lambda_C(z, X)$ and let $I_z = z\hat{z}$ be as in Lemma 4.1. Since

$$\|\hat{z}-u\| \geqslant \|\hat{z}-z\| - \|z-u\| > \frac{1}{\nu} p_C(\hat{z}-z) - \frac{r}{4}$$

$$= \frac{1}{2\nu} \lambda_C(z, X) - \frac{r}{4} > \frac{1}{2\nu l} - \frac{r}{4} \geqslant r_o - \frac{r}{4} \geqslant \frac{3}{4} r,$$

it follows that there is $y \in I_z$ such that $\|y-u\| = 3r/4$. Clearly $y \in E_o$ and

$$\|y-z\| \geqslant \|y-u\| - \|u-z\| > \frac{3}{4} r - \frac{r}{4} = \frac{r}{2}.$$

From the last inequality and (3.2) it follows that

$$r < \frac{2}{\mu} p_C(y-z). \tag{4.10}$$

Observe that

$$S(y, \alpha r) \subset S(u, r), \tag{4.11}$$

since, for arbitrary $v \in S(y, \alpha r)$ we have

$$\|v-u\| \leqslant \|v-y\| + \|y-u\| \leqslant \alpha r + \tfrac{3}{4} r \leqslant r.$$

Now we will prove that $S(y, \alpha r) \subset E \setminus E_{kl}$. It suffices to show that

$$S(y, \alpha r) \subset S(y, \rho_{yz}(\varepsilon_k)), \tag{4.12}$$

because if (4.12) is fulfilled we have $S(y, \alpha r) \subset E \setminus E_k \subset E \setminus E_{kl}$.

Clearly (4.10) and (4.8) imply

$$\alpha r < \frac{2\alpha}{\mu} p_C(y-z) \leqslant \frac{2\alpha_o}{1+2v} p_C(y-z). \tag{4.13}$$

Since δ_C^* is nondecreasing in the interval $[0, 1]$, by (4.13), (4.9), and the inequality $l > \lambda_C(z, X) - p_C(y-z) > 0$ we have

$$\delta_C^*\left(\frac{(1+2v)\,\alpha r}{2p_C(y-z)}\right) \leqslant \delta_C^*(\alpha_o) \leqslant \frac{\mu\varepsilon_k}{4l} \leqslant \frac{\mu\varepsilon_k}{4(\lambda_C(z, X) - p_C(y-z))}. \tag{4.14}$$

Moreover, since $r \leqslant r_0 \leqslant 1/2$, by (4.8) we have

$$\frac{2}{\mu}(1+2v)\,\alpha r \leqslant \frac{(1+2v)\,\alpha}{\mu} \leqslant \frac{\varepsilon_k}{2}. \tag{4.15}$$

From (3.4), (4.15), and (4.14) it follows that

$$\begin{aligned} &\omega_C(z, y; \lambda_C(z, X), (1+2v)\,\alpha r) \\ &\quad = \frac{2}{\mu}\left((1+2v)\,\alpha r + (\lambda_C(z, X) - p_C(y-z))\,\delta_C^*\left(\frac{(1+2v)\,\alpha r}{2p_C(y-z)}\right)\right) \\ &\quad \leqslant \frac{\varepsilon_k}{2} + \frac{\varepsilon_k}{2} = \varepsilon_k \end{aligned}$$

and, by virtue of Remark 3.1, we get

$$\alpha r \leqslant \frac{1}{1+2v}\,\omega_C^{-1}(z, y; \lambda_C(z, X), \varepsilon_k). \tag{4.16}$$

From (4.1), (4.16), (4.13), and the inequality $\alpha_o \leqslant 1/2$ it follows that $\alpha r \leqslant \rho_{yz}(\varepsilon_k)$. Thus (4.12) holds. Since u is arbitrary in E_{kl}, the inclusions (4.11) and (4.12) show that E_{kl} is porous in E. The proof of Theorem 4.1 is complete.

5. AMBIGUOUS LOCI

Let E and C be as in Section 1. For $X \in \mathfrak{B}$ consider the *ambiguous loci* of X with respect to C, i.e.,

$$A_C(X) = \{z \in E \mid \min_C(z, E) \text{ is not well posed}\}.$$

THEOREM 5.1. *Under the hypotheses of Theorem* 3.1 *if in addition E is separable then*

$$\mathfrak{B}^* = \{X \in \mathfrak{B} \mid A_C(X) \textit{ is dense in } E\}$$

is a residual subset of $\mathfrak{B}$.

Proof. For $a \in E$ and $r > 0$ define

$$\mathfrak{B}_{a,r} = \{X \in \mathfrak{B} \mid A_C(X) \cap S(a, r) = \varnothing\}.$$

We claim that $\mathfrak{B}_{a,r}$ is nowhere dense in $\mathfrak{B}$.

Indeed, let $X \in \mathfrak{B}_{a,r}$. Suppose that $a \notin X$ (if $a \in X$ we can take $\tilde{X} \in \mathfrak{B}$ near X such that $a \notin \tilde{X}$ and we use the argument below). Let $0 < \varepsilon < \min\{r, \lambda_C(a, X)\}$ and let $\varepsilon' = \min\{\varepsilon, \varepsilon\mu\}$. Since $X \in \mathfrak{B}_{a,r}$ there is $x_a \in X$ such that $P_C(x_a - a) = \lambda_C(a, X)$.

Define

$$y_1 = a + \left(\lambda_C(a, X) - \frac{\varepsilon'}{2}\right) \frac{x_a - a}{p_C(x_a - a)}.$$

Clearly $p_C(y_1 - a) = \lambda_C(a, X) - \varepsilon'/2 > \varepsilon/2$. Let $y_2 \in E$ be such that

$$p_C(y_2 - a) = p_C(y_1 - a) \qquad \text{and} \qquad \|y_2 - y_1\| = \frac{\varepsilon'}{2\nu}.$$

Define

$$y_i' = a + \frac{\varepsilon'(y_i - a)}{8p_C(y_i - a)}, \qquad i = 1, 2,$$

and

$$Y = X \cup \{y_1, y_2\}.$$

Since

$$\|x_a - y_1\| \leqslant \frac{1}{\mu} p_C(x_a - y_1) = \frac{\varepsilon'}{2\mu} \leqslant \frac{\varepsilon}{2}$$

and

$$\|x_a - y_2\| \leqslant \|x_a - y_1\| + \|y_1 - y_2\| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon'}{2\nu} \leqslant \varepsilon,$$

we have $h(Y, X) \leqslant \varepsilon$.

Define

$$\rho = \min \left\{ \frac{p_C(y_2 - y_1') - p_C(y_1 - y_1')}{2\nu}, \frac{p_C(y_1 - y_2') - p_C(y_2 - y_2')}{2\nu}, \frac{\varepsilon'}{6}, \frac{\varepsilon'}{6\nu} \right\}. \tag{5.1}$$

By virtue of Proposition 2.5 we have $\rho > 0$. Let $Z \in S_{\mathfrak{B}}(Y, \rho)$. (Here $S_{\mathfrak{B}}(Y, \rho)$ stands for the closed ball in $\mathfrak{B}$ with center Y and radius ρ.) Define $Z_i = Z \cap S(y_i, \rho)$. Observe that the sets Z_1 and Z_2 are nonempty and disjoint, because $\|y_1 - y_2\| \geqslant 3\rho$. Using Proposition 2.1 and 2.4 it is easy verify that

$$d(y_i, X) = \inf_{x \in X} \|x - y_i\| \geqslant \frac{1}{\nu} \inf_{x \in X} p_C(x - y_i)$$

$$\geqslant \frac{1}{\nu} p_C(x_a - y_1) = \frac{\varepsilon'}{2\nu} \geqslant 3\rho, \qquad i = 1, 2.$$

On the other hand

$$\|y_2' - y_1'\| = \frac{\varepsilon' \, \|y_2 - y_1\|}{8 p_C(y_1 - a)} < \frac{\|y_2 - y_1\|}{4} = \frac{\varepsilon'}{8\nu}.$$

From this and (3.2) it follows that

$$p_C(y_2' - y_1') < \frac{\varepsilon'}{8}.$$

Thus, for $z \in y_1' y_2'$ we have

$$\lambda_C(z, Z_i) \leqslant p_C(y_i' - z) + p_C(y_i - y_i') + \nu\rho$$

$$< \frac{\varepsilon'}{8} + \lambda_C(a, X) - \frac{5\varepsilon'}{8} + \nu\rho = \lambda_C(a, X) - \frac{\varepsilon'}{2} + \nu\rho \tag{5.2}$$

and

$$\lambda_C(z, X) \geqslant \lambda_C(a, X) - p_C(z-a) \geqslant \lambda_C(a, X) - \frac{\varepsilon'}{8}. \tag{5.3}$$

Since $\nu\rho - \varepsilon'/2 < -\nu\rho - \varepsilon'/8$ from (5.2) and (5.3) it follows that

$$\lambda_C(z, Z) = \lambda_C(z, Z_1 \cup Z_2) \qquad \text{for} \quad z \in y_1' y_2'. \tag{5.4}$$

On the other hand, using (5.1) we have

$$\lambda_C(y_1', Z_1) \leqslant p_C(y_1 - y_1') + \nu\rho \leqslant p_C(y_2 - y_1') - \nu\rho \leqslant \lambda_C(y_1', Z_2).$$

A similar argument shows that $\lambda_C(y_2', Z_2) \leqslant \lambda_C(y_2', Z_1)$. Since the map $\lambda(\cdot, Z_2) - \lambda_C(\cdot, Z_1)$ is continuous, nonnegative at y_1', and nonpositive at y_2', it follows that there is a $\tilde{z} \in y_1' y_2'$ such that

$$\lambda_C(\tilde{z}, Z_1) = \lambda_C(\tilde{z}, Z_2).$$

By (5.4) and the last equality we have

$$\lambda_C(\tilde{z}, Z) = \lambda_C(\tilde{z}, Z_1) = \lambda_C(\tilde{z}, Z_2).$$

Since $h(Z_1, Z_2) \geqslant \rho$, it follows that the minimization problem $\min_C(\tilde{z}, Z)$ is not well posed, and so $Z \notin \mathfrak{B}_{a,r}$. Since Z is arbitrary in $S_{\mathfrak{B}}(Y, \rho)$, we have $S_{\mathfrak{B}}(Y, \rho) \cap \mathfrak{B}_{a,r} = \varnothing$. Consequently $\mathfrak{B}_{a,r}$ is nowhere dense in $\mathfrak{B}$.

Let D be a countable dense subset of E and let Q_+ denote the set of all positive rationals. Define

$$\tilde{\mathfrak{B}} = \bigcup_{a \in D} \bigcup_{r \in Q_+} \mathfrak{B}_{a,r}.$$

Since $\tilde{\mathfrak{B}}$ is of the first Baire category in $\mathfrak{B}$, to complete the proof it suffices to show that $\mathfrak{B} \backslash \tilde{\mathfrak{B}} \subset \mathfrak{B}^*$. In fact, let $X \in \mathfrak{B} \backslash \tilde{\mathfrak{B}}$ and let $S(x, s)$ be an arbitrary ball in E. Take $a \in D$ and $r \in Q_+$ such that $S(a, r) \subset S(x, s)$. Since $X \notin \mathfrak{B}_{a,r}$, the set $A_C(X) \cap S(a, r)$ is nonempty, and so $X \in \mathfrak{B}^*$. This completes the proof of Theorem 5.1.

REFERENCES

1. E. Asplund, Chebyshev sets in Hilbert spaces, *Trans. Amer. Math. Soc.* **144** (1969), 236–240.
2. J. M. Borwein and S. Fitzpatrick, Existence of nearest points in Banach spaces, *Canad. J. Math.* **41** (1989), 702–720.
3. B. Brosowski and F. Deutsch, Some new continuity concepts for metric projections, *Bull. Amer. Math. Soc.* **78** (1972), 974–978.
4. E. V. Cheney, "Introduction to Approximation Theory," McGraw–Hill, New York, 1968.

5. F. S. De Blasi and J. Myjak, On almost well posed problems in the theory of best approximations, *Bull. Math. Soc. Sci. Math. R. S. Roumanie* (*N.S.*) **28** (1984), 109–117.
6. F. S. De Blasi and J. Myjak, Ambiguous loci of the nearest point mappings in Banach spaces, *Arch. Math.* **61** (1993), 377–386.
7. F. S. De Blasi, J. Myjak, and P. L. Papini, Porous sets in best approximation theory, *J. London Math. Soc.* **44** (1991), 135–142.
8. F. Deutsch, A general existence theorem for best approximations, *in* "Approximation in Theory and Praxis," pp. 71–82, Bibliographishes Institut, Mannheim, 1979.
9. M. Edelstein, A note on nearest points, *Quart. J. Math.* **21** (1969), 403–406.
10. M. Edelstein, On nearest points of sets in uniformly convex Banach spaces, *J. London Math. Soc.* **43** (1968), 375–377.
11. V. Klee, Remarks on nearest points in normal linear spaces, *in* "Proc. Coll. Convexity, Copenhagen, 1965," pp. 168–178, Univ. of Copenhagen, 1966.
12. V. Klee, E. Maluta, and C. Zanco, Tiling with smooth and rotund tiles, *Fund. Math.* **126** (1986), 269–290.
13. V. Klee, E. Maluta, and C. Zanco, Uniform properties of convex bodies, *Math. Ann.* **291** (1991), 153–177.
14. S. V. Konjagin, Approximation properties of arbitrary sets in Banach spaces, *Soviet Math. Dokl.* **19** (1978), 309–312.
15. S. V. Konjagin, On approximation properties of closed sets in Banach spaces and the characterization of strongly convex spaces, *Soviet Math. Dokl.* **251** (1980), 418–422.
16. K. S. Lau, Almost Chebyshev subsets in reflexive Banach spaces, *Indiana Univ. Math. J.* **27** (1978), 791–795.
17. I. Singer, The theory of best approximation and functional analysis, *in* "Regional Conference Series in Applied Mathematics," Vol. 13, SIAM, Philadelphia, 1971.
18. S. B. Stečkin, Approximation properties of sets in normed linear spaces, *Rev. Roumaine Math. Pures Appl.* **8** (1963), 5–13.
19. T. Zamfirescu, The nearest point mapping is single valued nearly everywhere, *Arch. Math.* **54** (1990), 563–566.
20. C. Zanco and A. Zucchi, Moduli of rotundity and smoothness for convex bodies, *Bolletino U.M.I.* **7–8** (1993), 833–855.